COMMON FIXED POINT THEOREMS FOR TWO SELFMAPS OF A COMPLETE D*- METRIC SPACE

Upender. S

Lecturer in Mathematics, Nagarjuna Government College (Autonomous), Nalgonda-508001, India

ABSTRACT

The purpose of this paper is to prove a common fixed point theorem for two selfmaps of a complete D-metric space. Also we show that a common fixed point theorem for two selfmaps of a metric space proved by Das and Naik ([5]) follows as a particular case of our result.*

Mathematics Subject Classification: *47H10, 54H25.*

Key Words: *D*-metric space, associated sequence, Fixed point theorem.*

INTRODUCTION AND PRELIMINARIES

The study of metric fixed point theorem has been researched extensively in the past decades, since fixed point theory plays a major role in Mathematics and Applied Sciences, such as Optimization, Mathematical Economics, Theory of Differential Equations, Mathematical Models and Potential Theory.

Different mathematicians tried to generalize the usual notion of metric space (X, d). In 1992 Dhage [2] has initiated the study of generalized metric space called D- metric space and fixed point theorems for selfmaps of such spaces. Later researchers have made a significant contribution to fixed point of D- metric spaces in [1], [3], and [4]. Unfortunately, almost all the fixed point theorems proved on D-metric spaces are not valid in view of papers [6], [7] and [8].

Recently Shaban Sedghi, Nabi Shobe and Haiyun Zhou [9], have introduced D*- metric spaces as a probable modification of D- metric spaces and proved some fixed point theorems.

Definition 1.1([9]): Let X be a non-empty set. A function $D^*: X^3 \to [0, \infty)$ is said to be a generalized metric or D*-metric or G-metric on X, if it satisfies the following conditions

(i) $D^*(x, y, z) \geq 0$ for all $x, y, z \in X$.
(ii) $D^*(x, y, z) = 0$ if and only if $x = y = z$.
(iii) $D^*(x, y, z) = D^*(\sigma(x, y, z))$ for all $x, y, z \in X$ where $\sigma(x, y, z)$ is any permutation of the set $\{x, y, z\}$.
(iv) $D^*(x, y, z) \leq D^*(x, y, w) + D^*(w, z, z)$ for all $x, y, z, w \in X$.

(IJISE) 2015, Vol. No. 1, Jan-Jun **e-ISSN: 2454-6402, p-ISSN: 2454-812X**

The pair (X, D *), where D * is a generalized metric on X is called a D*–metric space or a generalized metric space.

Example 1. 2: Let (X, d) be a metric space. Define D_1*: $X^3 \to [0, \infty)$ by

D_1*(x, y, z) = max {d(x, y), d(y, z), d (z, x)} for x, y, z ϵ X. Then (X, D_1*) is a generalized metric space.

Example 1.3: Let (X, d) be a metric space. Define D_2*: $X^3 \to [0, \infty)$ by

D_2*(x, y, z) = d(x, y) + d(y, z) + d (z, x) for x, y, z ϵ X. Then (X, D_2*) is a generalized metric space.

Example 1.4: Let X = R, define D*: $R^3 \to [0, \infty)$ by

$$D^*(x, y, z) = \begin{cases} 0 & \text{if } x = y = z \\ \max\{x, y, z\} & \text{otherwise} \end{cases}$$

Then (R, D *) is a generalized metric space.

Note 1.5: Using the inequality in (iv) and (ii) of Definition 1.1, one can prove that if (X, D *) is a D*–metric space, then

D *(x, x, y) = D *(x, y, y) for all x, y, ∈ X.

Infact D *(x, x, y) ≤ D *(x, x, x) + D *(x, y, y) = D *(x, y, y) and

D *(y , y , x) ≤ D *(y , y , y)+ D *(y , x , x) = D *(y , x , x), proving the inequity.

Definition1.6: Let (X, D*) be a D*-metric space. For x ϵ X and r > 0, the set B_{D^*}(x, r) = {y ϵ X; D*(x, y, y) < r} is called the open ball of radius r about x.

For example, if X = R and D*: $R^3 \to [0, \infty)$ is defined by

D*(x, y, z) = | x – y | + | y – z | +| z - x| for all x, y, z ϵ R. Then

B_{D^*}(0, 1) = {y ϵ R; D*(0, y, y) < 1}

= {y ϵ R; 2| y | < 1}

={y ϵ R; | y | < ½} = (- ½, ½).

Definition 1.7: Let (X, D *) be a D*–metric space and E ⊂ X.

(i) If for every x ∈ E, there is a δ > 0 such that B_{D^*} (x, δ)⊂ E, then E is said to be an open subset of X

(ii) If there is a k > 0 such that D *(x, y , y) < k for all x, y ∈E then E is said to be D*–bounded. It has been observed in [9] that, if τ is the set of all open sets in (X, D*), then τ is a topology on X (called the topology induced by the D*–metric) and also proved that B_{D^*} (x, r) is an open set for each x ∈ X and r > 0 ([9], Lemma 1.5). If (X, τ) is a compact topological space we shall call (X, D *) is a compact D*–metric space.

(IJISE) 2015, Vol. No. 1, Jan-Jun e-ISSN: 2454-6402, p-ISSN: 2454-812X

Definition1.8: Let (X, D *) be a D*–metric space. A sequence $\{x_n\}$ in X is said to

(i) converge to x if $D^*(x_n, x_n, x) = D^*(x_n, x, x) \to 0$ as $n \to \infty$

(ii) be a Cauchy sequence, if to each $\in > 0$, there is a natural number n_0 such that $D^*(x_n, x_n, x_m) < \in$ for all $m, n \geq n_0$.

It is easy to see (infact proved in [9], Lemma 1.8 and Lemma 1.9) that, if $\{x_n\}$ converges to x in(X, D*) then x is unique and that $\{x_n\}$ is a Cauchy sequence in (X , D*). However, a Cauchy sequence in a (X, D *) need not be convergent as shown in the example given below.

Example1.9: Let X = (0, 1] and $D^*(x, y, z) = | x - y | + | y - z | + | z - x |$ for $x, y, z \in X$, so that (X, D *) is a D*–metric space.

Define $x_n = \frac{1}{n}$ for n = 1, 2, 3 ……., then $D^*(x_n, x_n, x_m) = 2 | x_n - x_m | = 2 | \frac{1}{n} - \frac{1}{m} |$, so that

$D^*(x_n, x_n, x_m) \to 0$ as $m, n \to \infty$, proving $\{x_n\}$ is a Cauchy sequence in (X, D*).

Clearly $\{x_n\}$ does not converge to any point in X.

Definition 1.10: A D*–metric space (X, D *) is said to complete if every Cauchy sequence in it converges to some point in it.

It follows that the D*–metric space given in Example 1.9 is not complete.

Note 1.11: We have seen (In Example 1.2 and Example 1.3) that on any metric space (X, d) it is possible to define at least two D*–metrics, namely D_1^* and D_2^*, using the metric d. We shall call D_1^* and D_2^* as D*–metrics induced by d. Thus every metric space (X, d) gives rise to at least two D*–metric spaces (X, D_1^*) and (X, D_2^*). Also if (X, D *) is a D*–metric then defining $d_0(x, y) = D^*(x, y, y)$ for $x, y \in X$, we can show easily that (X, d_0) is a metric space and we shall call d_0 as a metric induced by D *.

The following result is of use for our discussion.

Theorem 1.12: Let (X, d) be a metric space and D_i^*(i =1, 2) be the two D*– metrics induced by d (given in Example 1.2 and Example 1.3). For any i (=1, 2) a sequence $\{x_n\}$ in (X, D_i^*) is a Cauchy sequence if and only if $\{x_n\}$ is a Cauchy sequence in (X , d).

Proof: - First note that for i =1, 2 we have

$d(x, y) \leq D_i^*(x, y, y) \leq 2d(x, y)$ for all $x, y \in X$.

Now the theorem follows immediately in view of the above inequality.

For example, if $\{x_n\}$ is a Cauchy sequence in (X, d), then for any given $\in > 0$ choose a natural number n_0 such that $m, n \geq n_0$ implies $d(x_m, x_n) < \frac{\in}{2}$; and note that for the same n_0 we have

$m, n \geq n_0$ implies $D_i{}^*(x_m, x_n, x_n) \leq 2d(x_m x_n) < \in$,

proving that $\{x_n\}$ is a Cauchy sequence in (X, D_i *).

Similarly, the other part of the theorem can be proved using the other inequality noted in the beginning of the proof.

Corollary1.13: Suppose (X, d) is a metric space. Let D_1 * and D_2 * be two D*– metrics induced by d, then for any i (=1, 2) the space (X, D_i *) is complete if and only if (X, d) is complete. Proof: - Follows from Theorem 1.12.

Definition 1.14: If (X,D*) is a D*-metric space, then D* is a continuous function on X^3, in the sense that $\lim_{n\to\infty} D^*(x_n,y_n,z_n) = D^*(x,y,z)$, whenever $\{(x_n,y_n,z_n)\}$ in X^3 converges to $(x, y, z) \in X^3$. Equivalently,

$$\lim_{n\to\infty} x_n = x,\ \lim_{n\to\infty} y_n = y,\ \lim_{n\to\infty} z_n = z \Leftrightarrow \lim_{n\to\infty} D^*(x_n,y_n,z_n) = D^*(x,y,z).$$

Notation: For any selfmap T of X, we denote T(x) by Tx.

If S and T are selfmaps of a set X, then any $z \in X$ such that Sz = Tz = z is called a common fixed point of S and T.

Two selfmaps S and T of X are said to be commutative if ST = TS where ST is their composition SoT defined by (SoT) x = STx for all $x \in X$.

Definition 1.15: Suppose S and T are selfmaps of a D*–metric space (X, D*) satisfying the condition $T(X) \subseteq S(X)$. Then for any $x_0 \in X$, $Tx_0 \in T(X)$ and hence $Tx_0 \in S(X)$, so that there is a $x_1 \in X$ with $Tx_0 = Sx_1$, since $T(X) \subseteq S(X)$. Now $Tx_1 \in T(X)$ and hence there is a $x_2 \in X$ with $Tx_2 \in T(X) \subseteq S(X)$ so that $Tx_1 = Sx_2$. Again $Tx_2 \in T(X)$ and hence $Tx_2 \in S(X)$ with $Tx_2 = Sx_3$. Thus repeating this process to each $x_0 \in X$, we get a sequence $\{x_n\}$ in X such that $Tx_n = Sx_{n+1}$ for $n \geq 0$. We shall call this sequence as an associated sequence of x_0 relative to the two selfmaps S and T. It may be noted that there may be more than one associated sequence for a point $x_0 \in X$ relative to selfmaps S and T.

Let S and T are selfmaps of a D*-metric space (X, D*) such that $T(X) \subseteq S(X)$. For any $x_o \in X$, if $\{x_n\}$ is a sequence in X such that $Tx_n = Sx_{n+1}$ for $n \geq 0$, then $\{x_n\}$ is called an associated sequence of x_0 relative to the two selfmaps S and T.

Definition 1.16: A function $\emptyset: [0,\infty) \to [0,\infty)$ is said to be a contractive modulus, if $\emptyset(0) = 0$ and $\emptyset(t) < t$ for $t > 0$.

(IJISE) 2015, Vol. No. 1, Jan-Jun **e-ISSN: 2454-6402, p-ISSN: 2454-812X**

Definition 1.17: A real valued function Ø defined on $X \subseteq R$ is said to be upper semi continuous, if $\lim_{n\to\infty} \sup \emptyset(t_n) \leq \emptyset(t)$ for every sequence $\{t_n\}$ in X with $t_n \to t$ as $n \to \infty$.

Definition 1.18: If S and T are selfmaps of a D*-metric space (X, D*) such that for every sequence $\{x_n\}$ in X with $\lim_{n\to\infty} Sx_n = \lim_{n\to\infty} Tx_n = t$, we have
$\lim_{n\to\infty} D^*(STx_n, TSx_n, TSx_n) = 0$, then we say that S and T are compatible

1. The Main Results:

2. 1 Introduction: If (X, D^*) is a complete D*- metric space and S, T are selfmaps satisfying certain conditions, we shall prove that, they have a common fixed point.

2.1.1 Theorem: Let S and T be selfmaps of a D*-metric space (X, D*) satisfying the conditions

(i) $T(X) \subseteq S(X)$

(ii) $D^*(Tx, Ty, Ty) \leq \emptyset(\xi(x, y))$ for all x, y ϵ X
where Ø is an upper semi continuous and contractive modulus and

(ii)' $\xi(x, y) = \max\{D^*(Sx, Sy, Sy), D^*(Sx, Tx, Tx), D^*(Sy, Ty, Ty), \frac{1}{2}[D^*(Sx, Ty, Ty) + D^*(Sy, Tx, Tx)]\}$

(iii)S is continuous
and

(iv) the pair (S, T) is compatible
Further, if

(iv)there is a point x_0 ϵ X and an associated sequence $\{x_n\}$ of x_0 relative to the two selfmaps such that the sequence $\{Tx_n\}$ and $\{Sx_n\}$ converge to some z ϵ X,
then z is the unique common fixed point for S and T.

Proof: From (v), we get

(2.1.2) $Sx_{2n}, Tx_{2n}, Sx_{2n+1}, Tx_{2n+1} \to z$ as $n \to \infty$.
Now, since S is continuous, we have, by (2.1.2)

(2.1.3) $S^2x_{2n+1} \to Sz$ and $STx_{2n+1} \to Sz$ as $n \to \infty$
Since the pair (S, T) is compatible, we have, in view of (2.1.1) that

(2.1.4) $\lim_{n\to\infty} D*(STx_{2n+1}, TSx_{2n+1}, TSx_{2n+1}) = 0$

(2.1.5) $TSx_{2n+1} \to Sz$ as $n \to \infty$.
Also from (ii), we have

(2.1.6) $D^*(TSx_{2n+1}, Tx_{2n}, Tx_{2n}) \leq \emptyset(\xi(Sx_{2n+1}, x_{2n}))$
where $\xi(Sx_{2n+1}, x_{2n}) = \max\{D^*(S^2x_{2n+1}, Sx_{2n}, Sx_{2n}), D^*(S^2x_{2n+1}, TSx_{2n+1}, TSx_{2n+1}), D^*(Sx_{2n}, Tx_{2n}, Tx_{2n}), \frac{1}{2}[D^*(S^2x_{2n+1}, Tx_{2n}, Tx_{2n}) + D^*(Sx_{2n}, TSx_{2n+1}, TSx_{2n+1})]\}$
which on letting n to ∞ and using the continuity of D*, gives

$\lim_{n\to\infty} \xi(Sx_{2n+1}, x_{2n})$ = max {D*(Sz, z, z), D*(Sz, Sz, Sz), D*(z, z, z), $\frac{1}{2}$ [D*(Sz, z, z) + D*(z, Sz, Sz)]}

=D*(Sz,z,z)

Therefore, letting n to ∞ in (2.1.6), and using the above we get

(2.1.7) D*(Sz, z, z) ≤ Ø(D*(Sz, z, z)).

Now, if Sz ≠ z, then D*(Sz, z, z) > 0 and by the definition of Ø, we get

Ø(D*(Sz, z, z)) < D*(Sz, z, z) contradicting (2.1.7)

Thus, we have Sz = z.

Now again from (ii) we have

(2.1.8) D*(Tz, Tx_{2n}, Tx_{2n}) ≤ Ø(ξ(z, x_{2n}))

where ξ(z, x_{2n}) = max {D*(Sz, Sx_{2n}, Sx_{2n}), D*(Sz, Tz, Tz),D*(Sx_{2n}, Tx_{2n} , Tx_{2n}),

$\frac{1}{2}$[D*(Sz, Tx_{2n}, T x_{2n})+ D*(Sx_{2n} , Tz, Tz)]}

in which on letting n to ∞, using Sz = z, the continuity of D* and the condition (v), we get

$\lim_{n\to\infty} \xi(z, x_{2n})$ = max {D*(Sz, z, z), D*(z, Tz, Tz), D*(z, z, z), $\frac{1}{2}$[D*(Sz, z, z) + D*(Sz, z, z)]}

= D*(z, Tz, Tz).

Again, letting n to ∞ in (2.1.8) and using the above, we get

(2.1.9) D*(Tz, z, z) ≤ Ø(D*(Tz, z, z))

and this will be a contradiction if Tz ≠ z, therefore Tz = z.

Thus 'z' is a common fixed point of S and T.

To prove that z is the unique common fixed point of S and T. If possible, suppose that z' is another common fixed point of S and T, then from (ii), we have

(2.1.10) D*(z, z', z') = D*(Tz, Tz', Tz') ≤ Ø(ξ(z, z')).

where ξ(z, z') = max {D*(Sz, z', z'), D*(z, Tz, Tz), D*(Sz' , Tz', Tz'),

$\frac{1}{2}$[D*(Sz, z', z') + D*(Sz', Tz, Tz)]}

= D*(z, z', z')

so that (2.1.10) gives

(2.1.11) D*(z, z', z') ≤ Ø (D*(z, z', z'))

and this will give a contradiction if z ≠ z'. Therefore z = z'. Thus, z is the unique common fixed point of S and T..

2.2 A Common Fixed Point Theorem for Two Selfmaps of a Complete D*- metric space

2.2.1 Theorem: Suppose S and T are selfmaps of a D*-metric space (X, D*) satisfying conditions (i) to (iv) of Theorem 2.1.1.

Further, if
(v)' (X, D*) is complete.
then S and T have unique common fixed point.

Before we give a proof of this, we prove the following lemma

2.2.2 Lemma: Let (X, D*) be a D*-metric space and S and T be selfmaps of X such that
(i) $T(X) \subseteq S(X)$
(ii) $D^*(Tx, Ty, Ty) \leq c\, \xi(x, y)$ for all $x, y \in X$
where $0 \leq c < 1$ and $\xi(x, y)$ is as defined in (ii)' of Theorem2.1.1

Further if
(iii) (X, D*) is complete
then for each $x_0 \in X$ and for any of its associated sequence $\{x_n\}$ relative to the selfmaps, the sequences $\{Tx_n\}$ and $\{Sx_n\}$ converges to same point $z \in X$.

Proof: Suppose S and T are selfmaps of a D*-metric space (X, D*) for which the conditions (i) and (ii) hold.

Let $x_0 \in X$ and $\{x_n\}$ be an associated sequence of x_0 relative to two selfmaps. Then, since $Tx_{2n} = Sx_{2n+1}$ and $Tx_{2n+1} = Sx_{2n+2}$ for $n \geq 0$.

Note that

$$\xi(x_{2n}, x_{2n+1}) = \max \{D^*(Sx_{2n}, Sx_{2n+1}, Sx_{2n+1}), D^*(Sx_{2n}, Tx_{2n}, Tx_{2n}), D^*(Sx_{2n+1}, Tn_{2n+1}, Tx_{2n+1}),\ \tfrac{1}{2}[D^*(Sx_{2n},Tx_{2n+1}, Tx_{2n+1}) + D^*(Sx_{2n+1}, Tx_{2n}, Tx_{2n})]\}$$

$$= \max \{D^*(Sx_{2n},Tx_{2n}, Tx_{2n}),\ D^*(Sx_{2n}, Tx_{2n}, Tx_{2n}),\ D^*(Tx_{2n}, Tx_{2n+1}, Tx_{2n+1}),\ \tfrac{1}{2}[D^*(Sx_{2n}, Tx_{2n+1}, Tx_{2n+1}) + D^*(Tx_{2n}, Tx_{2n}, Tx_{2n})]\}$$

$$= \max \{ D^*(Tx_{2n-1}, Tx_{2n}, Tx_{2n}),\ D^*(Tx_{2n}, Tx_{2n+1}, Tx_{2n+1}),\ \tfrac{1}{2} D^*(Tx_{2n-1}, Tx_{2n+1}, Tx_{2n+1})\}.$$

$$\xi(x_{2n}, x_{2n+1}) \leq \max \{ D^*(Tx_{2n-1},Tx_{2n},Tx_{2n}),\ D^*(Tx_{2n}, Tx_{2n+1}, Tx_{2n+1})\}$$

since

$$\tfrac{1}{2} D^*(Tx_{2n-1}, Tx_{2n+1}, Tx_{2n+1}) \leq \max \{D^*(Tx_{2n-1}, Tx_{2n}, Tx_{2n}),\ D^*(Tx_{2n}, Tx_{2n+1}, Tx_{2n+1})\}$$

Now, by (ii), $D^*(Tx_{2n}, Tx_{2n+1}, Tx_{2n+1}) \leq c\, .\, \xi(x_{2n}, x_{2n+1})$

$\leq c. \max\{D^*(Tx_{2n-1}, Tx_{2n}, Tx_{2n}), D^*(Tx_{2n}, Tx_{2n+1}, Tx_{2n+1})\}$

since $0 \leq c < 1$, it follows that the

$\max\{D^*(Tx_{2n-1}, Tx_{2n}, Tx_{2n}), D^*(Tx_{2n}, Tx_{2n+1}, Tx_{2n+1})\} = D^*(Tx_{2n-1}, Tx_{2n}, Tx_{2n})$

therefore $D^*(Tx_{2n}, Tx_{2n+1}, Tx_{2n+1}) \leq c. D^*(Tx_{2n-1}, Tx_{2n}, Tx_{2n})$ ……..(A)

Similarly, we can show that

$D^*(Tx_{2n-1}, Tx_{2n}, Tx_{2n}) \leq c. D^*(Tx_{2n-1}, Tx_{2n-2}, Tx_{2n-2})$ ……..(B)

From (A) and (B), we get

$$
\begin{aligned}
D^*(Tx_{2n}, Tx_{2n+1}, Tx_{2n+1}) &\leq c^2 \; D^*(Tx_{2n-1}, Tx_{2n-2}, Tx_{2n-2}) \\
&\leq c^4 \; D^*(Tx_{2n-3}, Tx_{2n-4}, Tx_{2n-4}) \\
&\quad - \;\; - - - - - - - - - - - - - - - - - - - \\
&\quad - \;\; - - - - - - - - - - - - - - - - - - - \\
&\leq c^{2n} \; D^*(Tx_1, Tx_0, Tx_0)
\end{aligned}
$$

Since $c^{2n} \to 0$ as $n \to \infty$ (because $c < 1$), the sequence $\{Tx_n\}$ is a Cauchy sequence in (X, D^*) and since it is complete, it converges to a point say $z \in X$.

Similarly we can prove that $\{Sx_{2n}\}$ converges to a point say $z' \in X$. Since $Sx_{2n+1} = Tx_{2n}$, we get $z = z'$. (In fact, $z' = \lim_{n\to\infty} S x_{2n+1} = \lim_{n\to\infty} T x_{2n} = z$), proving lemma.

2.2.3 Remark: The converse of Lemma 2.2.2 is not true. That is suppose S and T are selfmaps of a D*-metric space (X, D^*) satisfying condition (i) and (ii) of Lemma 2.2.2; even if for each $x_0 \in X$ and for each associated sequence $\{x_n\}$ of x_0 relative to S and T, the sequence $\{Sx_n\}$ and $\{Tx_n\}$ converges in X, then (X,D^*) need not complete. As an example, we have

2.2.4 Example: Let $X = [0, 1)$ and $d(x, y) = |x - y|$ for $x, y \in X$. Then we know that (X, d) is a metric space, which not complete.

Now, if $D^*(x, y, z) = \max\{d(x, y), d(y, z), d(z, x)\}$ for $x, y, z \in X$, then (X, D^*) is a D*-metric space and it is not complete.

Now, define selfmaps S and T of X, by

$$Sx = \frac{x+1}{6} \quad \text{if } x \in [0, 1).$$

and

$$T(x) = \begin{cases} \frac{1}{4}, & x = 0 \\ \frac{1}{5}, & x \in (0,1) \end{cases}$$

Then $T(X) = \{\frac{1}{5}, \frac{1}{4}\}$, while $S(X) = [\frac{1}{6}, \frac{2}{6})$ so that $T(X) \subseteq S(X)$ showing the condition (i). Now we prove the inequality (ii) considering various cases.

Case (i): Suppose $x = y = 0$ then

$D^*(Tx, Ty, Ty) = |\ \frac{1}{4} - \frac{1}{4}| = 0$

Obviously, the inequality (ii) holds for any c with $0 \le c < 1$.

Case (ii): Suppose x= 0 and y ≠ 0. Then

$D^*(Tx, Ty, Ty) = |\frac{1}{4} - \frac{1}{5}| = \frac{1}{20}$

$D^*(Sx, Sy, Sy) = |\frac{1}{6} - \frac{(y+1)}{6}| = \frac{y}{6}$

$D^*(Sx, Tx, Tx) = |\frac{1}{6} - \frac{1}{4}| = \frac{1}{12}$

$D^*(Sy, Ty, Ty) = |\frac{(y+1)}{6} - \frac{1}{5}| = \frac{|5y-1|}{30}$

$D^*(Sx, Ty, Ty) = |\frac{1}{6} - \frac{1}{5}| = \frac{1}{30}$

and

$D^*(Sy, Tx, Tx) = |\frac{(y+1)}{6} - \frac{1}{4}| = \frac{|2y-1|}{12}$

Therefore (ii) holds if

$\frac{1}{20} \le c. \max\{\frac{y}{6}, \frac{1}{12}, \frac{|5y-1|}{30}, \frac{1}{2}[\frac{1}{30} + \frac{|2y-1|}{12}]\}$

and this is possible by choosing c with $\frac{3}{10} \le c < 1$

Case (iii): Suppose x ≠ 0 and y = 0. This case is similar to Case (ii) with roles of x and y are interchanged with $\frac{3}{10} \le c \le 1$

Case (iv): Suppose x ≠ 0 and y ≠ 0. Then $D^*(Tx, Ty, Ty) = 0$ so that the inequality (ii)holds for any c with $0 \le c < 1$.

Now we prove that the sequence $\{Sx_n\}$ and $\{Tx_n\}$, converge to some $z \in X$ in the cases $x_0 = 0$ and $x_0 \neq 0$.

If $x_0 = 0$, then $Tx_0 = \frac{1}{4}$, since $T(X) \subseteq S(X)$, there is a $x_1 \in X$ with $Tx_0 = Sx_1$, that is $\frac{1}{4} = \frac{x_1+1}{6}$ or $x_1 = \frac{1}{2}$. Again there is a $x_2 \in X$ with $Tx_1 = Sx_2$, since $T(X) \subseteq S(X)$, that is $\frac{1}{5} = \frac{x_2+1}{6}$, this gives $x_2 = \frac{1}{5}$. Now there is a $x_3 \in X$ with $Tx_2 = Sx_3$, that is $\frac{1}{5} = \frac{x_3+1}{6}$ giving that $x_3 = \frac{1}{5}$. Also there is a $x_4 \in X$ with $Tx_3 = Sx_4$, since $T(X) \subseteq S(X)$, that is $\frac{1}{5} = \frac{x_4+1}{6}$, this gives $x_4 = \frac{1}{5}$. Showing that the sequence $\{Tx_n\}$ converges to a point $\frac{1}{5}$ in X. Similarly we can show that the sequence $\{Sx_n\}$ converges to $\frac{1}{5}$ in X.

Now, if $x_0 \neq 0$, then $Tx_0 = \frac{1}{5}$. Since $T(X) \subseteq S(X)$, there is a $x_1 \in X$ with $Tx_0 = Sx_1$, that is $\frac{1}{5} = \frac{x_1 + 1}{6}$ or $x_1 = \frac{1}{5}$. Again there is a $x_2 \in X$ with $Tx_1 = Sx_2$, since $T(X) \subseteq S(X)$, that is $\frac{1}{5} = \frac{x_2 + 1}{6}$, this gives $x_2 = \frac{1}{5}$. Now there is a $x_3 \in X$ with $Tx_2 = Sx_3$, that is $\frac{1}{5} = \frac{x_3 + 1}{6}$, giving that $x_3 = \frac{1}{5}$. Also there is a $x_4 \in X$ with $Tx_3 = Sx_4$, since $T(X) \subseteq S(X)$, that is $\frac{1}{5} = \frac{x_4 + 1}{6}$, which gives $x_4 = \frac{1}{5}$. Showing that the sequence $\{Tx_n\}$ is the constant sequence and converging to a point $\frac{1}{5}$ in X. Similarly we can show that the sequence $\{Sx_n\}$ converges to $\frac{1}{5}$ in X.

Corollary: Suppose S and T are selfmaps of a D*-metric space (X, D*) satisfying conditions (i) to (iv) of Theorem 2. 1. 1.

Further, if
$(v)^1$ (X, D*) is complete
then S and T have unique common fixed point.

Proof of Theorem 2.2.1:
In view of Lemma 2.2.2 the condition (v) of Theorem 2.1.1 holds in view of (v)'
Hence the corollary follows from Theorem 2.1.1.

2.2.5: Remark: Taking $\emptyset(t) = c\,t$ where $0 \leq c < 1$ in the Theorem 2.1.1, we get the following corollary immediately.

2.2.6: Corollary: Let S and T be selfmaps of a D*-metric space (X, D*) satisfying the conditions (i), (iii), (iv), (v) and
(ii)'' $D^*(Tx, Ty, Ty) \leq c\,\xi(x, y)$
where $\xi(x, y)$ is same as defined in (ii)' of Theorem 2.1.1. Then z is the unique common fixed point of S and T.

Now we show that a common fixed point theorem for two selfmaps of metric space proved by Das and Naik ([5]) follows as a particular case of our Theorem.

2.2.7 Corollary ([5], p.p 369-373): Let S and T be two selfmaps of a metric space (X, d) such that
(i) $T(X) \subseteq S(X)$
(ii) $d(Tx, Ty) \leq \emptyset(\eta(x, y))$ for all $x, y \in X$,
where $0 \leq c < 1$ and
(ii)' $\eta(x, y) = \max\{d(Sx, Sy), d(Sx, Tx), d(Sy, Ty), d(Sx, Ty), d(Sy, Tx)\}$
(iii) S is continuous,

and

(iv) ST = TS

Further, if

(v) X is complete

Then S and T have a unique common fixed point in X.

Proof: Given (X, d) is a metric space satisfying condition (i) to (v) of the corollary. If D_1*(x, y, z) = max {d(x, y), d(y, z), d (z, x)} then (X, D_1*) is a D*-metric space and D_1*(x, y, x) = d(x, y). Therefore (ii) can be written as D*(Tx, Ty, Ty) ≤ c. η(x, y) for all x, y ϵ X where

η(x, y) = max {D_1*(Sx, Sy, Sy), D_1*(Sx, Tx, Tx), D_1 *(Sy, Ty, Ty), D_1*(Sx, Ty, Ty), D_1*(Sy, Tx, Tx)} = ξ(x, y),

which is the same as condition (ii) of Theorem 2.2.1. Also since (X, d) is complete, we have (X, D_1*) is complete by Corollary1.13.

Now, S and T are selfmaps on (X, D_1*) satisfying conditions of Theorem 2.2.1 and hence the corollary follows.

REFERENCES

[1] Ahmad . B, Ashraf, M and Rhoades. B. E, "Fixed Point Theorems for Expansive Mappings in D- metric spaces" Indian Journal of Pure and Applied Mathematics Vol. 30, No. 10 pp1513-1518(2001)

[2] Dhage. B. C. "Generalised Metric Spaces and Mappings with Fixed Point" Bulletion of the Calcutta Mathematical Society, Vol. 84, No. 4, pp329-336(1992)

[3] Dhage. B. C. "A Common Fixed Point Principal in D- Metric Spaces" Bulletion of the Calcutta Mathematical Society, Vol. 91, No. 6, pp475-480(1999)

[4] Dhage B. C, Pathan A. M and Rhoads B. E, "A General Existence Principle for Fixed Point Theorems in D- metric spaces" International Journal of Mathematics and Mathematical Sciences" Vol. 23, No. 7 pp441-448(2000)

[5] Das. K. M and Vishwanth Naik. K., "Common Fixed Point Theorem for Commuting on a Metric Space", Proc. Amer. Math. Soc., 77 (1979); 369-373.

[6] Naidu S. V. R, Rao K. P. R and Srinivasa Rao N "On the Topology of D- metric spaces and Generalization of D- metric spaces from Metric Spaces" International Journal of Mathematics and Mathematical Sciences" Vol. 51, pp2719-2740(2004)

[7] Naidu S. V. R, Rao K. P. R and Srinivasa Rao N "On the Concepts of Balls in a D- metric spaces" International Journal of Mathematics and Mathematical Sciences" Vol. 1, pp133-141(2005)

[8] Naidu S. V. R, Rao K. P. R and Srinivasa Rao N "On Convergent Sequences and Fixed Point Theorems in D- metric spaces" International Journal of Mathematics and Mathematical Sciences" pp1969-1988(2005)

[9] Shaban Sedghi, Nabi Shobe and Haiyun Zhou "A Common Fixed Point Theorem in D*-metric spaces" Fixed Point Theory and Applications (2007)13 pages.